Induced functors on Drinfeld centers via monoidal adjunctions

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Outline

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Motivation — Morphisms of centers

Classical problem:

- $f\colon R \to S$ is a morphism of rings
- No restriction to a map $Z(R) \to Z(S)$ in general

Categorical analogues:

- Ring $(A, m, 1) \rightsquigarrow$ monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$
- Center $Z(A) \rightsquigarrow$ *Drinfeld center* $\mathcal{Z}(\mathcal{C})$
- Morphism of rings $\rightsquigarrow$ (strong) monoidal functor $G\colon \mathcal{C} \to \mathcal{D}$

$$\begin{tikzcd}
G(A) \otimes G(B) \arrow[rr, bend left, "\mathrm{lax}^G_{A,B}", "\sim"'] & & G(A \otimes B) \arrow[ll, bend left, "\mathrm{oplax}^G_{A,B}", "\sim"'] & + & \text{coherences}\ldots
\end{tikzcd}$$

- **Result (Flake–L.–Posur):** The right adjoint of G (often) induces a *braided lax monoidal* functor $\mathcal{Z}(R)\colon \mathcal{Z}(\mathcal{D}) \to \mathcal{Z}(\mathcal{C})$.

Some motivating examples

- $\phi\colon \mathsf{H} \hookrightarrow \mathsf{G}$ finite groups, $\omega \in H^3(\mathsf{G}, \Bbbk^\times)$ 3-cocycle,

$$\mathcal{Z}(\operatorname{Rep}\mathsf{H}) \to \mathcal{Z}(\operatorname{Rep}\mathsf{G}) \qquad \text{[Flake–Harman–L.]}$$

$$\mathcal{Z}(\mathbf{Vect}_{\mathsf{H}}^{\phi^*\omega}) \to \mathcal{Z}(\mathbf{Vect}_{\mathsf{G}}^{\omega}) \qquad \text{[Hannah–L.–Ros Camacho]}$$

  braided Frobenius monoidal functors

- Application: classifying connected étale algebras in $\mathcal{Z}(\mathbf{Vect}_{\mathsf{G}}^{\omega})$ [Davydov, Davydov–Simmons, L.–Walton, H.–L.–R.C.]
- For all $n \in \mathbb{Z}_{\geq 0}$, $t \in \mathbb{C}$,

$$\underline{\mathrm{Ind}}\colon \mathcal{Z}(\operatorname{Rep} S_n) \longrightarrow \mathcal{Z}(\underline{\operatorname{Rep}}\, S_t)$$

  braided Frobenius monoidal functor [Flake–Harman–L.]

- Application: classify indecomposable objects in $\mathcal{Z}(\underline{\operatorname{Rep}}\, S_t)$ [F.–H.–L.]

Goal: General results on induced functors on centers

Background — The Drinfeld center

$\mathcal{C}$ monoidal category, $\mathcal{M}$ a $\mathcal{C}$-bimodule

Definition ($\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$, Gelaki–Naidu–Nikshych, Greenough, . . .)

- **Objects:** (M, c), where $M \in \mathcal{M}$ and c *half-braiding*, a natural isomorphism $c^M_A \colon M \triangleleft A \xrightarrow{\sim} A \triangleright M$ satisfying:

$$c^M_{A \otimes B} = (A \triangleright c^M_B)(c^M_A \triangleleft B)$$

- **Morphisms:** $f \colon (M, c^M) \to (N, c^N) \overset{\text{corresponds to}}{\longleftrightarrow} f \in \mathrm{Hom}_{\mathcal{M}}(M, N)$ s.t.:

$$\begin{tikzcd}
M \triangleleft A \arrow[r, "c^M_A"] \arrow[d, "f \triangleleft A"] & A \triangleright M \arrow[d, "A \triangleright f"] \\
N \triangleleft A \arrow[r, "c^N_A"] & A \triangleright N.
\end{tikzcd}$$

Background — The Drinfeld center

Special cases:

- $\mathcal{C}^{\mathrm{reg}}$ — the *regular* $\mathcal{C}$-bimodule, action via $\otimes$
- $\mathcal{Z}(\mathcal{C}) := \mathcal{Z}_{\mathcal{C}}(\mathcal{C}^{\mathrm{reg}})$ is *braided monoidal* — the Drinfeld center of $\mathcal{C}$
- A *monoidal functor* $G\colon \mathcal{C} \to \mathcal{D}$ makes $\mathcal{D}$ a $\mathcal{C}$-bimodule, $\mathcal{D}^G$ — restricting $\mathcal{D}^{\mathrm{reg}}$ along G
- $\mathcal{Z}_{\mathcal{C}}(\mathcal{D}^G)$ is a *monoidal category* [Majid]

Proposition (2-Functoriality [Shimizu])

A $\mathcal{C}$-bimodule functor $F\colon \mathcal{M} \to \mathcal{N}$ induces a functor of categories

$$\mathcal{Z}_{\mathcal{C}}(F)\colon \mathcal{Z}_{\mathcal{C}}(\mathcal{M}) \to \mathcal{Z}_{\mathcal{C}}(\mathcal{N}).$$

Bimodule transformation $\eta\colon F \to G$ gives a natural transformation $\mathcal{Z}_{\mathcal{C}}(\eta)\colon \mathcal{Z}_{\mathcal{C}}(F) \to \mathcal{Z}_{\mathcal{C}}(G) \implies$ 2-functor $\mathcal{Z}_{\mathcal{C}}\colon \mathcal{C}\text{-}\mathbf{BiMod} \to \mathbf{Cat}$

Monoidal adjunctions

Define a **2-category** $\mathbf{Cat}^{\otimes}_{\mathrm{lax}}$:

- Objects: monoidal categories
- 1-Morphisms: *lax* monoidal functors
- 2-Morphisms: monoidal natural transformations $\eta \colon F \to G$:

$$\begin{tikzcd}
F(X) \otimes F(Y) \arrow[r, "\mathrm{lax}^F_{X,Y}"] \arrow[d, "\eta_X \otimes \eta_Y"] & F(X \otimes Y) \arrow[d, "\eta_{X \otimes Y}"] \\
G(X) \otimes G(Y) \arrow[r, "\mathrm{lax}^G_{X,Y}"] & G(X \otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{1} \arrow[dl, "\mathrm{lax}^G_0"'] \arrow[dr, "\mathrm{lax}^F_0"] & \\
F(\mathbb{1}) \arrow[rr, "\eta_{\mathbb{1}}"] & & G(\mathbb{1})
\end{tikzcd}$$

Definition (Monoidal adjunction)

A monoidal adjunction $G \dashv R$ is an adjunction *internal* to $\mathbf{Cat}^{\otimes}_{\mathrm{lax}}$.

- $G \dashv R$ monoidal adjunction $\Longrightarrow$ G is *strong* monoidal
- G strong monoidal $\Rightarrow$ $\exists!$ lax structure on R s.t. $G \dashv R$ is monoidal

The projection formula morphisms

Definition (Projection formula morphisms)

$$\begin{tikzcd} A \otimes RX \arrow[r, "\mathrm{unit}_A \otimes \mathrm{id}"'] \arrow[rr, bend left, "\mathrm{proj}^l_{A,X}"] & RG(A) \otimes RX \arrow[r, "\mathrm{lax}_{GA,X}"'] & R(GA \otimes X) \end{tikzcd}$$

If proj^l and proj^r are invertible, say: the *projection formula holds* for R.

- In representation theory (*Frobenius reciprocity*): $H \subset G$ finite groups, $\mathrm{Ind} \dashv \mathrm{Res}$ (op)monoidal adjunction,
$$\mathrm{proj}^l_{V,W} \colon \mathrm{Ind}^G_H(\mathrm{Res}^G_H(V) \otimes W) \xrightarrow{\sim} V \otimes \mathrm{Ind}^G_H(W)$$
- In algebraic geometry: $f \colon X \to Y$ morphism of schemes, $f^* \dashv f_*$, $\mathcal{E} \in \mathbf{QCoh}(Y)$, $\mathcal{F} \in \mathbf{QCoh}(X)$ locally free,
$$\mathrm{proj}^l_{\mathcal{E},\mathcal{F}} \colon \mathcal{E} \otimes_{\mathcal{O}_X} f_*(\mathcal{F}) \xrightarrow{\sim} f_*(f^*(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{F})$$

The projection formula morphisms

- If $\mathcal{C}, \mathcal{D}$ have finite products, they are monoidal categories with $\otimes = \times$, these are *cartesian closed* if $(-) \times A$ has a right adjoint $(-)^A$. A product preserving functor $G \colon \mathcal{C} \to \mathcal{D}$ with *left adjoint* gives a (op)monoidal adjunction $L \dashv G$.
- Such G is *cartesian closed*, i.e, $G(A^B) \simeq GA^{GB} \iff$
$$\mathrm{proj}^l_{A,X} = (\mathrm{counit}_A \, L\pi_A, L\pi_X) \colon L(GA \times X) \xrightarrow{\sim} A \times LX$$
is an isomorphism [Johnstone]

A sufficient criterion:

Proposition (Fausk–Hu–May, Flake–L.–Posur)

$\mathcal{C}$ *rigid (left and right duals exist)* $\Longrightarrow$ *the projection formula holds for* R

Categorical bimodule functors

Proposition (F.–L.–P.)

Let $G \dashv R$ be a monoidal adjunction.
projection formula $\Longrightarrow$ morphism of $\mathcal{C}$-bimodules $R\colon \mathcal{D}^G \to \mathcal{C}$ with:

$$\begin{tikzcd}
R(A \triangleright X) \arrow[r, "\mathrm{lin}^l_{A,X}"] \arrow[d, equal] & A \triangleright RX \arrow[d, equal] & R(X \triangleleft A) \arrow[r, "\mathrm{lin}^r_{X,A}"] \arrow[d, equal] & RX \triangleleft A \arrow[d, equal] \\
R(GA \otimes X) \arrow[r, "(\mathrm{proj}^l_{A,X})^{-1}"] & A \otimes RX & R(X \otimes GA) \arrow[r, "(\mathrm{proj}^r_{X,A})^{-1}"] & RX \otimes A
\end{tikzcd}$$

Monoidal adjunction: *Monoidal adjunctions* of $\mathcal{C}$-bimodules/categories:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "G"] \arrow[r, phantom, "\perp"] & \mathcal{D}^G \arrow[l, bend left, "R"] \arrow[r, phantom, "\Longrightarrow"] & \mathcal{Z}_{\mathcal{C}}(\mathcal{C}) \arrow[r, bend left, "\mathcal{Z}_{\mathcal{C}}(G)"] \arrow[r, phantom, "\perp"] & \mathcal{Z}_{\mathcal{C}}(\mathcal{D}^G) \arrow[l, bend left, "\mathcal{Z}_{\mathcal{C}}(R)"]
\end{tikzcd}$$

…since $\mathcal{Z}_{\mathcal{C}}\colon \mathcal{C}\text{-}\mathbf{BiMod} \to \mathbf{Cat}$ is a 2-functor

Functors on Drinfeld centers

We can now **compose**:

$$\begin{tikzcd}
\mathcal{Z}(\mathcal{D}) \arrow[rr, "\mathcal{Z}(R)"] \arrow[dr, bend right, "F^G"'] & & \mathcal{Z}_{\mathcal{C}}(\mathcal{C}) = \mathcal{Z}(\mathcal{C}) \\
& \mathcal{Z}_{\mathcal{C}}(\mathcal{D}^G) \arrow[ur, bend right, "\mathcal{Z}_{\mathcal{C}}(R)"'] &
\end{tikzcd}$$

$$F^G \colon \mathcal{Z}(\mathcal{D}) \hookrightarrow \mathcal{Z}(\mathcal{D}^G), \qquad (M, c^M) \mapsto (M, c^M_{G(-)})$$

Theorem (Flake–L.–Posur)

For a monoidal adjunction $G \dashv R$ satisfying the projection formula, R induces a braided lax monoidal functor $\mathcal{Z}(R) \colon \mathcal{Z}(\mathcal{D}) \to \mathcal{Z}(\mathcal{C})$, $(X, c) \mapsto (RX, c^R)$,

$$c^R_A = \left(RX \otimes A \xrightarrow{\mathrm{proj}^r_{X,A}} R(X \otimes GA) \xrightarrow{R(c_{GA})} R(GA \otimes X) \xrightarrow{(\mathrm{proj}^l_{A,X})^{-1}} A \otimes RX \right).$$

$$\mathrm{lax}^{\mathcal{Z}(R)}_{(X,c),(Y,d)} = \mathrm{lax}^R_{X,Y} \qquad \mathrm{lax}^{\mathcal{Z}(R)}_0 = \mathrm{lax}^R_0$$

Implication and Examples

Corollary

The functor $\mathcal{Z}(\mathcal{D}) \xrightarrow{\mathcal{Z}(R)} \mathcal{Z}(\mathcal{C})$ maps (commutative) monoids in $\mathcal{Z}(\mathcal{D})$ to (commutative) monoids in $\mathcal{Z}(\mathcal{C})$.

Example:

- $\mathsf{H} \subset \mathsf{G}$ finite groups, monoidal adjunction

$$\begin{tikzcd} \mathrm{Rep}(\mathsf{G}) \arrow[r, bend left, "\mathrm{Res}"] \arrow[r, phantom, "\perp"] & \mathrm{Res}(\mathsf{H}) \arrow[l, bend left, "\mathrm{CoInd} \simeq \mathrm{Ind}"] \end{tikzcd}$$

- $\mathcal{Z}(\mathrm{Rep}\,\mathsf{H}) \simeq {}^{\mathsf{H}}_{\mathsf{H}}\mathbf{YD}$ — Yetter–Drinfeld modules
  Objects: $V \in \mathrm{Rep}\,\mathsf{H}$ with coaction $\delta\colon V \to \mathsf{H} \otimes V$, $v \mapsto |v| \otimes v$, satisfying $|h \cdot v| = h|v|h^{-1}$
- Obtain braided lax monoidal functor $\mathcal{Z}(R)\colon {}^{\mathsf{H}}_{\mathsf{H}}\mathbf{YD} \to {}^{\mathsf{G}}_{\mathsf{G}}\mathbf{YD}$,

$$\mathcal{Z}(R)(V) = \mathsf{G} \otimes_{\mathsf{H}} V \qquad \text{with coaction} \qquad \delta^{\mathrm{Ind}}(g \otimes v) = g|v|g^{-1} \otimes v$$

Monoidal monads

Definition

A *monoidal monad* $T\colon \mathcal{C} \to \mathcal{C}$ is a monad in $\mathbf{Cat}^{\otimes}_{\mathrm{lax}}$.

This means:

- T is a monad
- $\mathcal{C}$ a monoidal category
- T comes equipped with a lax structure $\mathrm{lax}^T_{A,B}\colon T(A) \otimes T(B) \to T(A \otimes B)$
- $\mathrm{unit}^T_A\colon A \to T(A)$ and $\mathrm{mult}^T_A\colon T^2(A) \to T(A)$ are *monoidal* transformations

Lemma

$G \dashv R$ *monoidal adjunction* $\Longrightarrow T := RG$ *monoidal monad*

Commutative central monoids

Definition (Schauenburg . . .)

A **commutative central monoid** M in $\mathcal{C}$ is an commutative monoid (M, c^M) in $\mathcal{Z}(\mathcal{C})$. Structure: $\mathrm{mult}^M : M \otimes M \to M, \mathrm{unit}^M : \mathbb{1} \to M$.

- Now construct the monad

$$T_M : \mathcal{C} \to \mathcal{C}, \quad A \mapsto A \otimes M .$$

$$\mathrm{unit}^T_A := A \xrightarrow{A \otimes \mathrm{unit}^M} A \otimes M, \quad \mathrm{mult}^T_A := A \otimes M \otimes M \xrightarrow{A \otimes \mathrm{mult}^M} A \otimes M$$

- Lax structure: $\mathrm{lax}^T_0 := \mathbb{1} \xrightarrow{\mathrm{unit}^M} M$ and

$$\mathrm{lax}^T_{A,B} := A \otimes M \otimes B \otimes M \xrightarrow{A \otimes \mathrm{swap}_B \otimes M} A \otimes B \otimes M \otimes M \xrightarrow{A \otimes B \otimes \mathrm{mult}^M} A \otimes B \otimes M$$

Proposition

M *commutative central monoid* $\Longrightarrow$ T_M *monoidal monad*

Commutative central monoids

Another interpretation of the projection formula morphisms:

Proposition (F.–L.–P.)

Let $G \dashv R$ be a monoidal adjunction such that the projection formula holds.

(i) *Then $M := R(\mathbb{1})$ with*

$$c^{R\mathbb{1}} := R\mathbb{1} \otimes A \xrightarrow{\mathrm{proj}^r_{\mathbb{1},A}} RA \xrightarrow{(\mathrm{proj}^l_{A,\mathbb{1}})^{-1}} A \otimes R\mathbb{1}$$

$$\mathrm{mult} = \mathrm{lax}^R_{\mathbb{1},\mathbb{1}} \quad \textit{and} \quad \mathrm{unit} = \mathrm{lax}^R_0 \colon \mathbb{1} \to R\mathbb{1}$$

is a commutative central monoid in $\mathcal{C}$.

(ii) $T_M = (-) \otimes R(\mathbb{1}) \xrightarrow{\mathrm{proj}^l_{-,\mathbb{1}}} RG(-)$ *isomorphism of monoidal monads.*

Example: H $\subset$ G groups, monoidal adjunction $\mathrm{Res} \dashv \mathrm{CoInd}$.
$\Rightarrow R(\mathbb{1}) = \mathrm{CoInd}(\Bbbk) = \mathrm{Hom}_{\mathrm{Rep(H)}}(\Bbbk\mathrm{G}, \Bbbk) \cong \Bbbk(\mathrm{G}/\mathrm{H})$, the *algebra of functions* on G/H. **Note:** $\mathbf{Mod}_{\mathrm{Rep(G)}}\text{-}\Bbbk(\mathrm{G}/\mathrm{H}) \simeq \mathrm{Rep(H)}$ [Kirillov–Ostrik]

Monoidal Kleisli adjunctions

Assumption: $G \dashv R$ is an adjunction such that $RG = T$.

- Kleisli category $\mathbf{Klei}(T)$:
  - Same objects as $\mathcal{C}$
  - Morphisms $\mathrm{Hom}_{\mathbf{Klei}(T)}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, TB)$
- Diagram of functors:

$$\begin{tikzcd}
 & \mathcal{D} \arrow[dr, "R"] & \\
\mathcal{C} \arrow[ur, "G"] \arrow[dr, "\mathrm{Free}"'] & & \mathcal{C} \\
 & \mathbf{Klei}(T) \arrow[u, dashed, "\tilde{G}"] \arrow[ur, "\mathrm{Forg}"'] &
\end{tikzcd}$$

- T *monoidal monad* $\Longrightarrow$ $\mathbf{Klei}(T)$ *monoidal category*
  - Same tensor product $\otimes$ of objects as $\mathcal{C}$, same unit $\mathbb{1}$
  - tensor product of morphisms:

$$A \otimes C \xrightarrow{f \otimes g} TB \otimes TD \xrightarrow{\mathrm{lax}^{T}_{B,D}} T(B \otimes D) \in \mathrm{Hom}_{\mathbf{Klei}(T)}(A \otimes C, B \otimes D)$$

Monoidal Kleisli adjunctions

Theorem (Universal property, F.–L.–P.)

Assume T is a monoidal monad.

(i) *The adjunction* $$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathrm{Free}"] \arrow[r, phantom, "\perp"] & \mathbf{Klei}(T) \arrow[l, bend left, "\mathrm{Forg}"] \end{tikzcd}$$ *becomes a monoidal adjunction.*

(ii) $\mathrm{Free} \dashv \mathrm{Forg}$ *is the initial monoidal adjunction.*

Theorem (Characterization theorem, F.–L.–P.)

$G \dashv R$ *monoidal adjunction*

(i) *projection formula holds for* $R \Rightarrow$ *projection formula holds for* Forg

(ii) G *is also essentially surjective* $\Rightarrow \tilde{G} \colon \mathbf{Klei}(T) \to \mathcal{D}$ *monoidal equivalence*

Monoidal Kleisli adjunctions

Example:

- H finite-dimensional Hopf algebra, fiber functor $F\colon H\text{-}\mathbf{Mod} \to \mathbf{Vect}_{\Bbbk}$ is
  - (i) strong monoidal
  - (ii) essentially surjective, and
  - (iii) the projection formula holds (by rigidity).
- $R(\mathbb{1}) \cong H^*$ is a commutative central monoid.
- The characterization theorem implies:

$$\mathbf{Vect}_{\Bbbk} \simeq \mathbf{Klei}(T_{H^*}) \simeq \{\text{free } H^* \text{ modules in } H\text{-}\mathbf{Mod}\}$$

  $\Rightarrow$ *Fundamental theorem of Hopf modules*

- **More generally:** B finite-dimensional Hopf algebra object in ${}^{K}_{K}\mathbf{YD}$ set $H := B \rtimes K$ — Radford–Majid biproduct.
- $$\begin{tikzcd} H\text{-}\mathbf{Mod} \arrow[r, bend left, "\mathrm{Res}^H_K"] & K\text{-}\mathbf{Mod} \arrow[l, bend left, "\mathrm{CoInd}^H_K"] \end{tikzcd}$$ is a monoidal adjunction satisfying (i)–(iii).
- $K\text{-}\mathbf{Mod} \simeq \mathbf{Klei}(T_{B^*}) \simeq \{\text{free } B^*\text{-modules in } H\text{-}\mathbf{Mod}\}$

Monoidal Eilenberg–Moore categories

Idea: *free* modules (Kleisli) $\rightsquigarrow$ *all* modules (Eilenberg–Moore)

Recall: Projection formula for $G \dashv R$ $\Rightarrow$ isomorphism of *monoidal monads*:

$$RG \cong T_M = (-) \otimes M$$

for the commutative central monoid $M = R(\mathbb{1})$

Corollary

Eilenberg–Moore categories are given by $\mathbf{Mod}_{\mathcal{C}}\text{-}M$:

- *Objects: right M-modules internal to $\mathcal{C}$*
- *Morphisms: morphisms in $\mathcal{C}$ commuting with M-action*

General construction of monoidal structure on Eilenberg–Moore category [Seal]

Monoidal Eilenberg–Moore categories

Assumption: $\mathcal{C}$ has *reflexive coequalizers* and $\otimes$ preserves them in both components

Theorem (Pareigis, Schauenburg, . . .)

$\mathbf{Mod}_{\mathcal{C}}\text{-}M$ *is monoidal with relative tensor product*

$$\begin{tikzcd} A \otimes M \otimes B \arrow[r, shift left, "\mathrm{act}^A \otimes B"] \arrow[r, shift right, "(A \otimes \mathrm{act}^B) c^M_B"'] & A \otimes B \arrow[r, "\mathrm{quo}_{A,B}"] & A \otimes_M B, \end{tikzcd}$$

Consequence: Monoidal Eilenberg–Moore adjunction:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathrm{Free}"] \arrow[r, phantom, "\perp"] & \mathbf{Mod}_{\mathcal{C}}\text{-}M \arrow[l, bend left, "\mathrm{Forg}"] \end{tikzcd}$$

Monoidal Eilenberg–Moore categories

Assumption: $\mathcal{C}$ has *reflexive coequalizers* and $\otimes$ preserves them in both components.

Theorem (Universal property, F.–L.–P.)

$G \dashv R$ *monoidal adjunction*, *projection formula holds for* R

(i) *The projection formula holds for* Forg

(ii) *There is a unique induced lax monoidal functor* $\tilde{R}$*:*

$$\begin{tikzcd}
& \mathcal{D} \arrow[dd, dashed, "\tilde{R}"] \arrow[dr, "R"] & \\
\mathcal{C} \arrow[ur, "G"] \arrow[dr, "\mathrm{Free}"'] & & \mathcal{C} \\
& \mathbf{Mod}\text{-}\mathcal{C}M \arrow[ur, "\mathrm{Forg}"'] &
\end{tikzcd}$$

(iii) $\mathrm{Free} \dashv \mathrm{Forg}$ *is the terminal monoidal adjunction.*

Note: We can derive a crude monoidal monadicity theorem

Local modules

Definition (Local Modules [Pareigis])

(M, c^M) commutative central monoid

$$\mathbf{Mod}^{\mathrm{loc}}_{\mathcal{Z}(\mathcal{C})}\text{-}M \subseteq \mathbf{Mod}_{\mathcal{Z}(\mathcal{C})}\text{-}M$$

Full subcategory on *local* M-modules (A, act^A), i.e.:

$$\mathrm{act}^A \, \Psi_{M,A} \Psi_{A,M} = \mathrm{act}^A \, .$$

$\mathbf{Mod}^{\mathrm{loc}}_{\mathcal{Z}(\mathcal{C})}\text{-}M$ is braided monoidal [Pareigis]

Theorem (Schauenburg)

There is an equivalence of braided monoidal categories

$$\mathcal{Z}(\mathbf{Mod}_{\mathcal{C}}\text{-}M) \simeq \mathbf{Mod}^{\mathrm{loc}}_{\mathcal{Z}(\mathcal{C})}\text{-}M.$$

Local modules

Assumption: $\mathcal{C}$ has *reflexive coequalizers* and $\otimes$ preserves them in both components.

We can recognize the induced functor $\mathcal{Z}(\mathrm{Forg})$ on Drinfeld centers:

Corollary (F.–L.–P.)

$\mathrm{Forg}\colon \mathbf{Mod}_{\mathcal{C}}\text{-}M \to \mathcal{C}$ *induces braided lax monoidal functor*

$$\mathcal{Z}(\mathrm{Forg})\colon \mathcal{Z}(\mathbf{Mod}_{\mathcal{C}}\text{-}M) \to \mathcal{Z}(\mathcal{C})$$

Schauenburg's equivalence implies $\mathcal{Z}(\mathrm{Forg})$ *corresponds to*

$$\mathrm{Forg}^{\mathrm{loc}}\colon \mathbf{Mod}^{\mathrm{loc}}_{\mathcal{Z}(\mathcal{C})}\text{-}M \to \mathcal{Z}(\mathcal{C})$$

Lax monoidal structure: the coequalizer morphism $A \otimes B \to A \otimes_M B$.

Functors of Yetter–Drinfeld modules

Application: functors of Yetter–Drinfeld categories over Hopf algebras.

- $\varphi\colon K \to H$ morphism of Hopf algebras:
- $$\begin{tikzcd} H\text{-}\mathbf{Mod} \arrow[r, bend left, "\mathrm{Res}_{\varphi}"] \arrow[r, phantom, "\perp"] & K\text{-}\mathbf{Mod} \arrow[l, bend left, "\mathrm{CoInd}_{\varphi}"] \end{tikzcd} \qquad \begin{tikzcd} K\text{-}\mathbf{Comod} \arrow[r, bend left, "\mathrm{Res}^{\varphi}"] \arrow[r, phantom, "\perp"] & H\text{-}\mathbf{Comod} \arrow[l, bend left, "\mathrm{Ind}^{\varphi}"] \end{tikzcd}$$
- Comodule induction $\mathrm{Ind}^{\varphi}(V) = H\square_K V$ *always* satisfies the projection formula
- Module coinduction $\mathrm{CoInd}^{\varphi}(V) = \mathrm{Hom}_K(H, V)$ satisfies the projection formula if H is *finitely-generated projective* as a left K-module.
- induced functors:

$$\mathcal{Z}(\mathrm{CoInd}_{\varphi})\colon {}^{K}_{K}\mathbf{YD} \to {}^{H}_{H}\mathbf{YD} \quad \text{or} \quad \mathcal{Z}(\mathrm{Ind}^{\varphi})\colon {}^{H}_{H}\mathbf{YD} \to {}^{K}_{K}\mathbf{YD}$$

Examples

- Morphism of affine algebraic groups $\phi\colon \mathsf{K} \to \mathsf{G}$ (morphism of Hopf algebras $\varphi = \phi^*\colon \mathcal{O}_{\mathsf{G}} \to \mathcal{O}_{\mathsf{K}}$)
  - Braided lax monoidal functor
$$\mathcal{Z}(\mathrm{Ind}^{\phi^*})\colon \mathbf{QCoh}(\mathsf{K}/^{\mathrm{ad}}\mathsf{K}) \to \mathbf{QCoh}(\mathsf{G}/^{\mathrm{ad}}\mathsf{G}).$$
  - $\mathcal{Z}(\mathrm{Rep}\,\mathsf{G}) = \mathcal{Z}(\mathcal{O}_{\mathsf{G}}\text{-}\mathbf{Comod}) \simeq \mathbf{QCoh}(\mathsf{G}/^{\mathrm{ad}}\mathsf{G})$, quasi-coherent sheaves on the quotient stack $\mathsf{G}/^{\mathrm{ad}}\mathsf{G}$
  - Convolution tensor product
- Kac–De Concini quantum group $U_\epsilon(\mathfrak{g})$ (odd root of unity ϵ)
  - *central* Hopf subalgebra $\mathcal{O}_{\mathsf{H}}$, for $\mathsf{H} = (\mathsf{N}^- \times \mathsf{N}^+) \rtimes \mathsf{T}$
  - Inclusion $\iota\colon \mathcal{O}_{\mathsf{H}} \hookrightarrow U_\epsilon(\mathfrak{g})$
  - induces a braided lax monoidal functor
$$\mathcal{Z}(\mathrm{CoInd}_\iota)\colon \mathbf{QCoh}(\mathsf{H}/^{\mathrm{ad}}\mathsf{H}) \longrightarrow {}^{U_\epsilon(\mathfrak{g})}_{U_\epsilon(\mathfrak{g})}\mathbf{YD},$$
  - Image of $\mathbb{1} = \Bbbk$: central commutative monoidal $u_\epsilon(\mathfrak{g})^* \cong \mathrm{CoInd}_\iota(\Bbbk)$ over $U_\epsilon(\mathfrak{g})$

Outlook — Frobenius monoidal functors

- Monoidal ambiadjunctions $F \dashv G \dashv F$
- Left and right adjoint F both gives two projection formula morphisms
- If these are mutual inverses, the $\mathcal{Z}(F)$ is a Frobenius monoidal functor
- Hopf algebra case: New concept of Frobenius monoidal extension of Hopf algebras

. . . Thank you for your attention!